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Locally finite spaces and the join operator

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Abstract The importance of digital geometry in image processing is well documented. To understand global properties of digital spaces and manifolds we need a solid understanding of local properties. We shall study the join operator, which combines two topological spaces into a new space. Under the natural assumption of local finiteness, we show that spaces can be uniquely decomposed as a join of indecomposable spaces.

Keywords: join operator, Alexandrov space, smallest-neighborhood space, locally finite space.

1. Introduction

Topological properties of digital images play an important role in image processing. Much of the theoretical development has been motivated by the needs in applications, for example digital Jordan curve theorems and the theory of digitization. A classical survey is [11] by Kong and Rosenfeld.

Inspired by the new mathematical objects that have emerged from this process, mathematicians have started to study digital geometry from a more theoretical perspective, developing the theories in different directions. Evako et al. [5, 6] considered, for example, n-dimensional digital surfaces satisfying certain axioms. These surfaces were later considered by Daragon et al. [4]. Khalimsky spaces as surfaces, embedded in spaces of higher dimension have been studied the author in [15].

We shall study, not digital spaces, but a tool that can be used in such a study: the join operator. Evako used an operation on directed graphs called the join to aid the analysis. We will generalize this construction and study its properties. In particular we show how a space can be decomposed into indecomposable pieces put together by the join operator. We will give conditions for uniqueness of this decomposition.

2. Digital spaces and the Khalimsky topology

We present here a mathematical background. The purpose is primarily to introduce notation and formulate some results that we will need. A reader not familiar with these concepts is recommended to take a look at, for example, Kiselman's [9] lecture notes.

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2.1 Topology and smallest-neighborhood spaces

Not all topological spaces are reasonable digital spaces, but the class of finite topological spaces is too small. It does not include $\mathbb{Z}^n$.

In every topological space, a *finite* intersection of open sets is open, whereas an *arbitrary* intersection of open sets need not be open. If the space is finite, however, there are only finitely many open sets, so finite spaces fulfill a stronger requirement: arbitrary intersections of open sets are open.

Alexandrov [1] considers topological spaces, finite or not, that fulfill this stronger requirement. We shall call such spaces *smallest-neighborhood spaces*. Another name that is often used is *Alexandrov spaces*, but this name has one disadvantage: it has already been used for spaces appearing in differential geometry.

Let B be a subset of a topological space X. The *closure* of B is the intersection of all closed sets containing B. The closure is usually denoted by $\overline{B}$. We shall instead write $\mathscr{C}_X(B)$ for the closure of B in X. This notation allows us to specify in what space we consider the closure and is also a notation dual to $\mathscr{N}_X$ defined below.

Using the same B and X as above, we define $\mathscr{N}_X(B)$ to be the intersection of all open sets containing B. In general $\mathscr{N}_X(B)$ is not an open set, but in a smallest-neighborhood space it is; $\mathscr{N}_X(B)$ is the smallest neighborhood containing B. If there is no danger of ambiguity, we will just write $\mathscr{N}(B)$ and $\mathscr{C}(B)$ instead of $\mathscr{N}_X(B)$ and $\mathscr{C}_X(B)$. If x is a point in X, we define $\mathscr{N}(x) = \mathscr{N}(\{x\})$ and $\mathscr{C}(x) = \mathscr{C}(\{x\})$. Note that $y \in \mathscr{N}(x)$ if and only if $x \in \mathscr{C}(y)$.

We have already seen that $\mathscr{N}(x)$ is the smallest neighborhood of x. Conversely, if every point of the space has a smallest neighborhood, then an arbitrary intersection of open sets is open; hence this existence could have been used as an alternative definition of a smallest-neighborhood space.

A point x is called *open* if $\mathscr{N}(x) = \{x\}$, that is, if $\{x\}$ is open. The point is called *closed* if $\mathscr{C}(x) = \{x\}$. If x is either open or closed it is called *pure*, otherwise it is called *mixed*.

Adjacency and connectedness

A topological space X is called *connected* if the only sets, which are both closed and open, are the empty set and X itself. A *connectivity component* (sometimes called a "connected component") of a topological space is a connected subspace which is maximal with respect to inclusion.

Two distinct points x and y in X are called *adjacent* if the subspace $\{x, y\}$ is connected. It is easy to check that x and y are adjacent if and only $y \in \mathscr{N}(x)$ or $x \in \mathscr{N}(y)$. Another equivalent condition is $y \in \mathscr{N}(x) \cup \mathscr{C}(x)$. The *adjacency neighborhood* of a point x in X is denoted $\mathrm{AN}_X(x)$ and is the set $\mathscr{N}_X(x) \cup \mathscr{C}_X(x)$. It is practical also to have a notation for the set of points adjacent to a point, but not including it. Therefore the *adjacency set*

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in X of a point x, denoted $\mathscr{A}_X(x)$, is defined to be $\mathscr{A}_X(x) = \mathrm{AN}_X(x) \smallsetminus \{x\}$. Often, we just write $\mathscr{A}(x)$ and $\mathrm{AN}(x)$.

A point adjacent to x is sometimes called a *neighbor* of x. This terminology, however, is somewhat dangerous since a neighbor of x need not be in the smallest neighborhood of x.

Separation axioms

Kolmogorov's separation axiom, also called the T_0 axiom, states that given two distinct points x and y, there is an open set containing one of them but not the other. An equivalent formulation is that $\mathscr{N}(x) = \mathscr{N}(y)$ implies $x = y$ for every x and y. The $T_{1/2}$ axiom states that all points are pure. Clearly any $T_{1/2}$ space is also T_0.

The next separation axiom is the T_1 axiom. It states that points are closed. In a smallest-neighborhood space this implies that every set is closed and hence that every set is open. Therefore, a smallest-neighborhood space satisfying the T_1 axiom must have the discrete topology, and thus, is not very interesting.

Duality

Since the open and closed sets in a smallest neighborhood space X satisfy exactly the same axioms, there is a complete symmetry. Instead of calling the open sets open, we may call them closed, and call the closed sets open. Then we get a new smallest-neighborhood space, called the *dual* of X, which we will denote by X'.

The Alexandrov–Birkhoff preorder

There is a correspondence between smallest-neighborhood spaces and partially preordered sets. Let X be a smallest-neighborhood space and define $x \preccurlyeq y$ to hold if $y \in \mathscr{N}(x)$. We shall call this relation the *Alexandrov–Birkhoff preorder*. It was studied independently by Alexandrov [1] and by Birkhoff [3].

The Alexandrov–Birkhoff preorder is always reflexive (for all x is $x \preccurlyeq x$) and transitive (for all $x, y, z \in X$, $x \preccurlyeq y$ and $y \preccurlyeq z$ imply $x \preccurlyeq z$). A relation satisfying these conditions is called a *preorder* (or *quasiorder*).

A preorder is an *order* if it in addition is anti-symmetric (for all $x, y \in X$, $x \preccurlyeq y$ and $y \preccurlyeq x$ imply $x = y$). The Alexandrov–Birkhoff preorder is an order if and only the space is T_0. In conclusion, it would therefore be possible to formulate the results of this paper in the language of orders instead of the language of topology.

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2.2 The Khalimsky topology

We shall construct a connected topology on $\mathbb{Z}$, which was introduced by Efim Khalimsky (see Khalimsky et al. [8] and references there).

If m is an odd integer, let $P_m =]m-1, m+1[$, and if m is an even integer, let $P_m = \{m\}$. The family $\{P_m\}_{m\in\mathbb{Z}}$ forms a partition of the Euclidean line $\mathbb{R}$ and thus we may consider the quotient space. If we identify each P_m with the integer it contains, we get the *Khalimsky topology* on $\mathbb{Z}$. We call this space the *Khalimsky line*. Since $\mathbb{R}$ is connected, the Khalimsky line is a connected space.

It follows readily that an even point is closed and that an odd point is open. In terms of smallest neighborhoods, we have $\mathscr{N}(m) = \{m\}$ if m is odd and $\mathscr{N}(n) = \{n-1, n, n+1\}$ if n is even.

Perhaps this topology should instead be called the Alexandrov–Hopf–Khalimsky topology, since it appeared in an exercise [2, I:Paragraph 1:Exercice 4]. However, it was Khalimsky who realized that this topology was useful in connection with digital geometry and studied it systematically. Since this topology is also called the Khalimsky topology in the literature, we will keep this name.

A different approach to digital spaces, using cellular complexes, was introduced independently by Herman and Webster [7] and by Kovalevsky [13]. The results of this article apply to such spaces, since they are topologically equivalent to spaces of the type introduced in this article. See, for example, Klette [10].

Khalimsky intervals and arcs

Let a and b, $a \leqslant b$, be integers. A *Khalimsky interval* is an interval $[a, b]\cap\mathbb{Z}$ of integers with the topology induced from the Khalimsky line. We will denote such an interval by $[a, b]_{\mathbb{Z}}$ and call a and b its *endpoints*. A *Khalimsky arc* in a topological space X is a subspace that is homeomorphic to a Khalimsky interval. If any two points in X are the endpoints of a Khalimsky arc, we say that X is *Khalimsky arc-connected*.

Theorem 1. *A T_0 smallest-neighborhood space is connected if and only if it is Khalimsky arc-connected.*

A proof can be found in [14, Theorem 11]. Slightly weaker is [8, Theorem 3.2c]. The theorem also follows from Lemma 20(b) in [12].

Let us define the *length* of a Khalimsky arc, A, to be the number of points in A minus one, $L(A) = \operatorname{card} A - 1$. If a smallest-neighborhood space X is T_0 and connected, Theorem 1 guarantees that length of the shortest arc connecting x and y in X is a finite number. This observation allows us to define a metric ρ_X on X, which we call the *arc metric*.

$$\rho_X(x, y) = \min(L(A);\ A \subset X \text{ is a Khalimsky arc containing } x \text{ and } y).$$

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The arc metric is defined on X, but it is important to bear in mind that the topology of X is not the metric topology defined by ρ_X. The metric topology is of course the discrete topology.

Examples of smallest-neighborhood spaces

We conclude this section with a few examples to indicate that the class of smallest-neighborhood spaces and spaces based on the Khalimsky topology is sufficiently rich to be worth studying.

Example 1. The *Khalimsky plane* is the Cartesian product of two Khalimsky lines and in general, *Khalimsky n-space* is $\mathbb{Z}^n$ with the product topology. Points with all coordinates even are closed and points with all coordinates odd are open. Points with both even and odd coordinates are mixed. It is easy to check that $\mathscr{A}(p) = \{x \in \mathbb{Z}^n;\ \|p - x\|_\infty = 1\}$ if p is pure.

Example 2. We may consider a quotient space $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ for some even integer $m \geqslant 2$. Such a space is called a *Khalimsky circle*. If $m \geqslant 4$, $\mathbb{Z}_m$ is a compact space that is locally homeomorphic to the Khalimsky line. (If m were odd, we would identify open and closed points, resulting in a space with the indiscrete or chaotic topology, i.e., where the only open sets are the empty set and the space itself.)

3. Locally finite and locally countable spaces

A topological space actually stored in a computer is finite. Nevertheless, it is of theoretical importance to be able to treat infinite spaces, like $\mathbb{Z}^2$, since the existence of a boundary often tends to complicate matters.

On the other hand, spaces where a points can have an infinite number of neighbors seems less likely to appear in computer applications. In this situation, not even the local information can be stored.

Definition 1. A smallest-neighborhood space is called *locally finite* if every point in it has a finite adjacency neighborhood. If every point has a countable adjacency neighborhood, it is called *locally countable*.

We need to assume the axiom of choice (in fact the countable axiom of choice, which states that from a *countable* collection of nonempty sets we can select one element from each set, is sufficient) to prove the following.

Proposition 1. *Let X be a locally finite space. If X is connected, then X is countable.*

Proof. If X is not T_0, we consider the quotient space $\tilde{X}$, where points with identical neighborhoods have been identified. $\tilde{X}$ is T_0 and X is countable if $\tilde{X}$ is countable. It is therefore sufficient to prove the result for T_0 spaces.

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Let x be any point in X. By induction based on local finiteness, the ball $B_n(x) = \{y; \in X; \rho_X(x,y) \leqslant n\}$, where ρ_X is the arc-metric on X (here we use that the space is T_0), is finite for every $n \in \mathbb{N}$. Since $X = \bigcup_{n=0}^{\infty} B_n(x)$, the result is true, since the countable axiom of choice implies that a countable union of finite sets is countable. □

In a similar way, we can also characterize countable smallest-neighborhood spaces.

Proposition 2. *A smallest-neighborhood space X is countable if and only if it is locally countable and has countably many connectivity components.*

Proof. It is clear that a countable space is locally countable and has countably many components. The countable axiom of choice implies that countable unions of countable sets are countable. So if X is connected and locally countable, a slightly modified version of the proof of Proposition 1 shows that X is countable. If X has countably many connectivity components and each component is countable, then X is countable. □

The following proposition states that the set of open points is dense in a locally finite space (and by duality a corresponding result holds for the closed points). It is obvious that the open points form the smallest dense set.

Proposition 3. *Let X be a smallest-neighborhood space and let $S \subset X$ be the set of open points in X and T be the set of closed points. If X is T_0 and locally finite, then $X = \mathscr{C}(S) = \mathscr{N}(T)$.*

Proof. We will prove the fist equality, the other follows by duality. Let y_0 be any point in X. Let $Y_0 = \mathscr{N}(y_0)$. If Y_0 is a singleton set, then $y_0 \in S$. Otherwise, for $k \geqslant 0$, choose $y_{k+1} \in Y_k \smallsetminus \{y_k\}$ and let $Y_{k+1} = \mathscr{N}(y_{k+1})$. Clearly $Y_{k+1} \subset Y_k$ and since X is T_0, it follows that $y_k \notin Y_{k+1}$. Repeat the construction above recursively until Y_k is a singleton set, at most $\mathrm{card}(Y_0)-1$ steps are needed. Note that y_k is an open point and that $y_0 \in \mathscr{C}(y_k)$. Since y_0 was arbitrarily chosen, we are done. □

The relation in the proposition need not hold if the space is not locally finite. Consider the space $\mathbb{Z}$ where the (non-trivial) open sets are given by intervals $]-\infty, m]_{\mathbb{Z}}$, $m \in \mathbb{Z}$. This space contains no open point.

On the other hand, if we add the point $-\infty$ and declare the open sets to be intervals $[-\infty, m]_{\mathbb{Z}}$, where $m \in \mathbb{Z} \cup \{-\infty\}$, then the point $-\infty$ is open and the whole space equals $\mathscr{C}(-\infty)$. Thus, finite neighborhoods are not necessary for the conclusion of the proposition.

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4. The join operator

A well-known way to combine two topological spaces, X and Y is to take the coproduct (disjoint union), $X \coprod Y$. The pieces, X and Y, are completely independent in this construction. We shall introduce another way of combining two spaces, namely the join operator.

Definition 2. Let X and Y be two topological spaces. The *join* of X and Y, denoted $X \vee Y$, is a topological space over the disjoint set union of X and Y, where a subset $A \subset X \dot{\cup} Y$ is declared to be open if either

(i) $A \cap X$ is open in X and $A \cap Y = \emptyset$, or

(ii) $A \cap X = X$ and $A \cap Y$ is open in Y.

Note that a set $B \subset X \vee Y$ is closed if and only if

(i) $B \cap X$ is closed in X and $B \cap Y = Y$, or

(ii) $B \cap X = \emptyset$ and $B \cap Y$ is closed in Y.

While this definition makes sense for any topological space, it is a strange operation on large spaces. For example, the join of the real line and the circle, $\mathbb{R} \vee S^1$, is a compact space, which is T_0 but not T_1. In fact, $X \vee Y$ cannot be T_1 unless X or Y is empty.

From now on, we shall only consider the join of smallest-neighborhood spaces. In this case, the definition boils down to the following. If X and Y are smallest-neighborhood spaces, then the topology of $X \vee Y$ is given by $\mathscr{N}_{X \vee Y}(x) = \mathscr{N}_X(x)$ if $x \in X$ and $\mathscr{N}_{X \vee Y}(y) = \mathscr{N}_Y(y) \cup X$ if $y \in Y$. Apparently, the join of two smallest-neighborhood spaces is a smallest-neighborhood space. This definition of the join is compatible with the join of directed graphs, see [6, p. 111]. In terms of the Alexandrov–Birkhoff preorder, every element of X is declared to be larger than any element of Y; X is placed on top of Y in $X \vee Y$. This motivates also our notation $X \vee Y$. Formally, the order, i.e., pairs (x, y) satisfying $x \succcurlyeq y$, on $X \vee Y$, which we here denote $\operatorname{Ord}(X \vee Y)$ is

$$\operatorname{Ord}(X \vee Y) = \operatorname{Ord}(X) \cup \operatorname{Ord}(Y) \cup X \times Y.$$

The join of two connected locally finite spaces need not be locally finite. In fact, the join of two spaces is locally finite if and only if the spaces are finite and locally countable if and only if both are countable. In view of Proposition 2, the join of two connected and locally countable spaces is locally countable. When p is a point, we write $p \vee X$ instead of $\{p\} \vee X$. Here $\{p\}$ is the topological space with one point.

4.1 Basic properties

The following three properties in the next proposition are straightforward to prove.

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Proposition 4. *The join operator has the following properties for all smallest-neighborhood spaces X, Y and Z.*
(i) $X = \emptyset \vee X = X \vee \emptyset$ *(has a unity).*
(ii) $(X \vee Y) \vee Z = X \vee (Y \vee Z)$ *(is associative).*
(iii) $(X \vee Y)' = Y' \vee X'$.

The following proposition lists some topological properties.

Proposition 5. *Let X and Y be smallest-neighborhood spaces.*
(i) *$X \vee Y$ is T_0 if and only if X and Y are T_0.*
(ii) *$X \vee Y$ is compact if and only if Y is compact.*
(iii) *If $X \neq \emptyset$ and $Y \neq \emptyset$ then $X \vee Y$ is connected.*

Proof. To prove (i), note that X is open in $X \vee Y$. If $x \in X$ and $y \in Y$, then X is an open set containing x but not y. It follows that $X \vee Y$ can fail to be T_0 only for a pair of points in X or a pair of points in Y. But in this case the equivalence is obvious.

Next we prove (ii). Assume first that Y is not compact and that $\{A_i\}_{i \in I}$ is an open cover of Y without a finite subcover. Let $B_i = A_i \cup X$ for each $i \in I$. Then $\{B_i\}_{i \in I}$ is an open cover of $X \vee Y$ without a finite subcover. For the other direction, assume that Y is compact and take an open cover, $\{B_i\}_{i \in I}$, of $X \vee Y$. By restriction, it induces an open cover of Y with elements $B_i \cap Y$. But this cover has a finite subcover, $\{B_i \cap Y\}_{i=1}^n$, since Y is compact. It follows that $\{B_i\}_{i=1}^n$ is finite subcover of $X \vee Y$ since any B_i where $B_i \cap Y \neq \emptyset$ covers X.

To prove (iii), assume that $x \in X$ and $y \in Y$. Since $x \in \mathscr{N}(y)$, it is clear that x and y are adjacent. If $a, b \in X$, then $\{a, y, b\}$ a connected set for the same reason. In the same way $\{c, x, d\}$ is connected if $c, d \in Y$. □

In fact all properties of the two proceeding propositions, except (iii) of Proposition 4, are true for general topological spaces, not only for smallest-neighborhood spaces.

4.2 Decomposable spaces

If $Z = X \vee Y$ implies $X = \emptyset$ or $Y = \emptyset$, then the smallest-neighborhood space Z is called *indecomposable*, otherwise Z is called *decomposable*. Note that a locally finite decomposable smallest-neighborhood space is in fact finite.

Proposition 6. *If a smallest-neighborhood space Z is decomposable, then for every $x, y \in Z$ there is a point $z \in Z$ such that $x, y \in \mathrm{AN}(z)$. (If Z is T_0, an equivalent and more comprehensible condition is that $\rho_Z(x, y) \leqslant 2$.)*

Proof. The result is given by the proof of Proposition 5, Part (iii). □

The converse implication is not true, as the following example shows.

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Example 3. Let $X = \{a_1, a_2, b, c_1, c_2\}$ be a set with 5 points, and equip it with a topology as follows: $\mathscr{N}(a_1) = \{a_1\}$, $\mathscr{N}(a_2) = \{a_2\}$, $\mathscr{N}(b) = \{a_1, b\}$, $\mathscr{N}(c_1) = \{a_1, a_2, b, c_1\}$, and $\mathscr{N}(c_2) = \{a_1, a_2, c_2\}$. It is easy to see that $\rho(x, y) \leqslant 2$ for any $x, y \in X$.

Suppose that X were decomposable, $X = A \vee C$. We would necessarily have $a_1, a_2 \in A$ since these points are open, and $c_1, c_2 \in C$ since these points are closed. But b cannot be in A since $b \notin \mathscr{N}(c_2)$ and b cannot be in C since $a_2 \notin \mathscr{N}(b)$. Therefore, X is indecomposable.

We have the following uniqueness result for the decomposition.

Theorem 2. *Let X be a smallest-neighborhood space. If $X = Y \vee Z$ and $X = \tilde{Y} \vee \tilde{Z}$, where Y and $\tilde{Y}$ are indecomposable and non-empty, then $Y = \tilde{Y}$ and $Z = \tilde{Z}$.*

Proof. It is sufficient to prove that $Y = \tilde{Y}$. If $Y \neq \tilde{Y}$, we may suppose that $Y \smallsetminus \tilde{Y}$ is not empty and let $p \in Y \smallsetminus \tilde{Y}$. Then $\tilde{Y} \subset Y$, for if there were a point q in $\tilde{Y} \smallsetminus Y$, then $q \in Z$ so that $\mathscr{C}_X(q) \subset Z$, since $X = Y \vee Z$. But by assumption $p \notin \tilde{Y}$, so therefore $p \in \tilde{Z}$. As $q \in \tilde{Y}$ and $X = \tilde{Y} \vee \tilde{Z}$, this implies $p \in \mathscr{C}_X(q) \subset Z$, which is a contradiction since $p \in Y$.

Define $B = Y \smallsetminus \tilde{Y}$, which is non-empty by assumption. Take two arbitrary points $a \in \tilde{Y}$ and $b \in B$. Note that $b \in \tilde{Z}$. Hence $a \in \mathscr{N}_X(b)$ and thus also $a \in \mathscr{N}_Y(b)$. It follows that $\mathscr{N}_Y(b) = \mathscr{N}_B(b) \cup \tilde{Y}$. If $\tilde{Y}$ and B are equipped with the relative topology, we have $Y = \tilde{Y} \vee B$, so Y is decomposable contrary to the assumption. □

If a smallest-neighborhood space X is locally finite, then repeated use of Theorem 2 together with associativity gives the following.

Corollary 1. *If X is a locally finite smallest-neighborhood space, then X can be written in a unique way as $X = Y_1 \vee \cdots \vee Y_n$ where each Y_i is indecomposable and non-empty.*

Note that if X is decomposable so that $n > 1$, then X is necessarily finite. While Theorem 2 is quite general, there is no universal cancellation law; if $A \vee X = B \vee X$ we cannot conclude that A and B are homeomorphic (which we will denote by $A \simeq B$), as the following example shows.

Example 4. Let N denote the set of natural numbers equipped with the topology given by $\mathscr{N}(n) = \{i \in \mathbb{N};\ i \geqslant n\}$. For a positive integer m, let $N_m = N \cap [0, m]$, with the induced topology. It follows that $N = N_m \vee N$ for every m, but N_m is homeomorphic to N_k only if $m = k$.

On the other hand, if X is locally finite, the unique decomposition of Corollary 1 implies that both a right and a left cancellation law hold. In fact, we may weaken the hypothesis slightly.

Theorem 3. *Let X be a smallest-neighborhood space. Suppose there are finitely many locally finite spaces $Y_1, \ldots, Y_n$ so that $X = Y_1 \vee \cdots \vee Y_n$. Then for all smallest-neighborhood spaces A and B we have*

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(i) $X \vee A \simeq X \vee B$ *implies* $A \simeq B$,

(ii) $A \vee X \simeq B \vee X$ *implies* $A \simeq B$.

Note that X is locally finite only if $n = 1$ or if every Y_i (and hence X itself) is finite.

Proof. We shall prove (i). The second claim follows by duality. Let Y be any smallest-neighborhood space. Define an *opening chain* in Y to be a finite sequence of pairwise distinct points $(y_0, \ldots, y_n)$ in Y such that $y_{i+1} \in \mathscr{N}_Y(y_i)$ for $0 \leqslant i \leqslant n$. The number n is the called the *length* of the open chain. We say that the chain *starts* in y_0. Let $h_Y : Y \to \mathbb{N} \cup \{\infty\}$ be defined by

$$h_Y(y) = \sup(n;\ \text{there is an opening chain in } Y \text{ of length } n \text{ starting in } y).$$

From the construction of X, it is straightforward to check that $h_{A \vee X}(x)$ is a finite number for any $x \in X$. Furthermore, for any $a \in A$ we have

$$h_{X \vee A}(a) \geqslant 1 + \sup_{x \in X} h_{X \vee A}(x) \tag{1}$$

since every $x \in X$ belongs to $\mathscr{N}_{X \vee A}(a)$. Furthermore,

$$\sup_{x \in X} h_{X \vee A}(x) = \sup_{x \in X} h_{X \vee B}(x), \tag{2}$$

since $h_X(x) = h_{X \vee Y}(x)$ for any space Y and any $x \in X$.

Let $\phi \colon X \vee A \to X \vee B$ be a homeomorphism. A chain is mapped to a chain by ϕ, so we have the identity $h_{X \vee A}(x) = h_{X \vee B}(\phi(x))$ for every $x \in X \vee A$. In view of (1) and (2) this implies that

$$h_{X \vee B}(\phi(a)) \geqslant 1 + \sup_{x \in X} h_{X \vee B}(x),$$

for every $a \in A$. Hence $\phi(X) = X$ and $\phi(A) = B$, which proves (i). □

5. Applications

We shall demonstrate how the tools we have developed can be used to give a simple proof of a known result in digital topology, namely the characterization of neighborhoods in Khalimsky spaces (Evako et al. [6]). We start with a consequence of Proposition 6.

Corollary 2. *Let $p \in \mathbb{Z}^n$ be pure. Then $\mathscr{A}(p)$ is indecomposable. If p is closed,* $\mathrm{AN}(p) = \mathscr{A}(p) \vee p$ *and if p is open,* $\mathrm{AN}(p) = p \vee \mathscr{A}(p)$.

Proof. For the first property, notice that if $q \in \mathscr{A}(p)$ then $r = 2p - q$ (as vectors in $\mathbb{Z}^n \subset \mathbb{R}^n$) also belongs to $\mathscr{A}(p)$. It is readily checked that $\rho_{\mathscr{A}(p)}(q, r) = 4$ if $n \geqslant 2$ (if $n = 1$, $\mathscr{A}_{\mathbb{Z}}(p)$ is not connected and the result immediate). Proposition 6 shows that $\mathscr{A}(p)$ is indecomposable. The decomposition of $\mathrm{AN}(p)$ is straightforward. □

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If p is pure, it is easy to explicitly describe $\mathrm{AN}(p)$. We have

$$\mathrm{AN}(p) = \{x \in \mathbb{Z}^n;\ \|x - p\|_\infty \leqslant 1\},$$

so that p is adjacent to $3^n - 1$ points.

More generally, let $q \in \mathbb{Z}$ be any point with j even coordinates and k odd coordinates. To simplify our notation, we note that there is a homeomorphism of $\mathbb{Z}^n$, build from a translation $q \mapsto q + v$, where $v \in 2\mathbb{Z}^n$, and permutation of coordinates, which takes q to the point $\tilde{q} = (0, \dots, 0, 1, \dots 1) \in \mathbb{Z}^n$, where there are j zeros and k ones (and $j + k = n$).

It follows that

$$\mathscr{N}(\tilde{q}) = [-1, 1]^j \times \{1\}^k$$

and that

$$\mathscr{C}(\tilde{q}) = \{0\}^j \times [0, 2]^k.$$

Since $\mathrm{AN}(\tilde{q}) = \mathscr{N}(\tilde{q}) \cup \mathscr{C}(\tilde{q})$, we see that q is adjacent to $3^j + 3^k - 2$ points.

Let $\mathbf{0}_j$ denote the point $(0, \dots, 0) \in \mathbb{Z}^j$ and let $\mathbf{1}_k = (1, \dots, 1) \in \mathbb{Z}^k$. It is easy to see that $\mathscr{N}(\tilde{q}) \simeq \mathscr{N}_{\mathbb{Z}^j}(\mathbf{0}_j)$ and it follows that $\mathscr{N}(\tilde{q}) \smallsetminus \{\tilde{q}\}$ is homeomorphic to $\mathscr{A}_{\mathbb{Z}^j}(\mathbf{0}_j)$, which is indecomposable by Corollary 2. By a similar argument, $\mathscr{C}(q) \smallsetminus \{\tilde{q}\}$ is homeomorphic to $\mathscr{A}_{\mathbb{Z}^k}(\mathbf{1}_k)$. Note that $\mathscr{A}(\mathbf{0}_0) = \mathscr{A}(\mathbf{1}_0) = \emptyset$.

It is straightforward to check that in any smallest-neighborhood space X and for any $x \in X$ we have

$$\mathscr{A}(x) \simeq (\mathscr{N}(x) \smallsetminus \{x\}) \vee (\mathscr{C}(x) \smallsetminus \{x\})$$

and we obtain the following.

Proposition 7. *Let $q \in \mathbb{Z}^n$. Then*

$$\mathscr{A}_{\mathbb{Z}^n}(q) \simeq \mathscr{A}_{\mathbb{Z}^j}(\mathbf{0}_j) \vee \mathscr{A}_{\mathbb{Z}^k}(\mathbf{1}_k),$$

where j is the number of even coordinates in q and k is the number of odd coordinates.

As stated from the outset, this result is known, and serves only as an illustration of the formalism introduced.

6. Conclusion

We have studied the join operator, which takes two topological spaces and combines them into a new space. This operation is interesting primarily for small spaces, viz. adjacency neighborhoods. We have seen that if we assume local finiteness of the spaces involved, we can show that a space is decomposed in a unique way into indecomposable spaces, and we have given a criterion to recognize indecomposable spaces.

The machinery can be used to systematically investigate local properties of digital topological spaces. Hopefully, this will lead to new insights into the nature of such spaces.

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