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General approach for fuzzy mathematical morphology

Antony T. Popov

Faculty of Mathematics and Informatics, University of Sofia "St. Kliment Ohridski", Bulgaria
atpopov@fmi.uni-sofia.bg

Abstract This work shows how a generalized approach for constructing dilation-erosion adjunctions on fuzzy sets can be defined using appropriate chosen complete lattice. Some applications in the field of computation with uncertainties are given, more precisely in the interval arithmetic and the calculations with fuzzy numbers. Applications to image segmentation such as geodesic operations and reconstruction are given as well. Also we discuss how intuitionistic fuzzy sets can be used as structuring elements for fuzzy morphological operations, especially in fuzzy hit-or-miss transform. The aim is to find objects with close to given shape and size on digital images.

Keywords: complete lattice, fuzzy sets, interval and arithmetic operation, fuzzy arithmetic operation, hit-or-miss transform, T-invariant operation, geodesic operation.

1. Introduction

There are several approaches for fuzzifying mathematical morphology, see for instance [1]. In our work we step on the framework of Deng and Heijmans (see for details [3]) based on conjunctors-implicators adjoint fuzzy logical operators. We generalize this definition presenting a universal framework and we define naturally fuzzy geodesic morphological operations. Also, this model is applied to fuzzy arithmetic, built by analog to the interval arithmetic [8] which makes possible the definition of inner addition and multiplication of fuzzy numbers. On the other hand, in the pioneering works of Serra [11] and Heijmans [5] it is demonstrated that the hit-or-miss transform plays an essential role in shape analysis. So, here we define general fuzzy hit-or-miss transform for grey-scale image segmentation and we show how it is related to the theory of intuitionistic fuzzy sets (IFS).

In this work we use the same notions and notations about complete lattices and the morphological operations on them as in [5]. For instance, let $\mathcal{L}$ be a complete lattice with a supremum generating family l, and let T be an Abelian group of automorphisms of $\mathcal{L}$ acting transitively over l. The elements of T are denoted by τ_x, namely for any $x \in l$, $\tau_x(o) = x$, where o

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is a fixed element of l interpreted as an origin. Then also, we can consider a symmetry in $\mathcal{L}$ as $\check{A} = \bigvee_{a\in l(A)} \tau_a^{-1}(o)$. Evidently $\check{a} = \tau_a^{-1}(o) = \tau_{\check{a}}(o)$ for any $a \in l$. If A is an arbitrary element of the lattice $\mathcal{L}$ let us denote by $l(A) = \{a \in l \mid a \leq A\}$ the supremum-generating set of A. Following [5] we define the operations

$$\delta_A \;=\; \bigvee_{a\in l(A)} \tau_a \tag{1}$$

and

$$\varepsilon_A \;=\; \bigwedge_{a\in l(A)} \tau_a^{-1} \;=\; \bigwedge_{a\in l(A)} \tau_{\check{a}}, \tag{2}$$

which form an adjunction. δ_A and ε_A are T-invariant operators called *dilation and erosion by the structuring element* A. Remind that a pair of operators (ε, δ) between two lattices, $\varepsilon : \mathcal{M} \mapsto \mathcal{L}$ and $\delta : \mathcal{L} \mapsto \mathcal{M}$ is called an *adjunction* if for every two elements $X \in \mathcal{L}$ and $Y \in \mathcal{M}$ it follows that

$$\delta(X) \leq Y \iff X \leq \varepsilon(Y).$$

In [5] it is proved that if (ε, δ) is an adjunction then ε is erosion and δ is dilation. In other hand, every dilation $\delta : \mathcal{L} \mapsto \mathcal{M}$ has a unique adjoint erosion $\varepsilon : \mathcal{M} \mapsto \mathcal{L}$, and vice-versa.

2. Fuzzy sets and fuzzy morphological operations

Consider the set E called the universal set. A fuzzy subset A of the universal set E can be considered as a function $\mu_A : E \mapsto [0, 1]$, called the membership function of A. $\mu_A(x)$ is called the degree of membership of the point x to the set A. The ordinary subsets of E, sometimes called 'crisp sets', can be considered as a particular case of a fuzzy set with membership function taking only the values 0 and 1.

Let $0 < \alpha \leq 1$. An α-cut of the set X (denoted by $[X]_\alpha$) is the set of points x, for which $\mu_X(x) \geq \alpha$.

The usual set-theoretical operations can be defined naturally on fuzzy sets: Union and intersection of a collection of fuzzy sets is defined as supremum, resp. infimum of their membership functions. Also, we say that $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ for all $x \in E$. The complement of A is the set A^c with membership function $\mu_{A^c}(x) = 1 - \mu_A(x)$ for all $x \in E$. If the universal set E is linear, like the d-dimensional Euclidean vector space $\mathbb{R}^d$ or the space of integer vectors with length d, then any geometrical transformation arising from a point mapping can be generalised from sets to fuzzy sets by taking the formula of this transformation for graphs of numerical functions, i.e., for any transformation ψ like scaling, translation, rotation etc. we have that $\psi(\mu_A(x)) = \mu_A(\psi^{-1}(x))$. Therefore we can transform fuzzy sets by transforming their $\alpha-$cuts like ordinary sets. Further on, for simplicity, we shall write simply $A(x)$ instead of $\mu_A(x)$.

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Say that the function $c(x,y) : [0,1] \times [0,1] \mapsto [0,1]$ is conjunctor conjunctor if c is increasing in the both arguments, $c(0,1) = c(1,0) = 0$, and $c(1,1) = 1$. We say that a conjunctor is a *t-norm* if it is commutative, i.e. $c(x,y) = c(y,x)$, associative $c(c(x,y),z) = c(x,c(y,z))$ and $c(x,1) = x$ for every number $x \in [0,1]$, see for instance [1, 6].

Say that the function $i(x,y) : [0,1] \times [0,1] \mapsto [0,1]$ is implicator implicator if i is increasing in y and decreasing in x, $i(0,0) = i(1,1) = 1$, and $i(1,0) = 0$.

In [3] a number of adjoint conjunctor-implicator pairs are proposed. Here we give examples of two of them:

$$c(b,y) = \min(b,y),$$
$$i(b,x) = \begin{cases} x & x < b, \\ 1 & x \geq b. \end{cases}$$

$$c(b,y) = \max(0, b + y - 1),$$
$$i(b,x) = \min(1, x - b + 1).$$

The first pair is known as operations of Gödel-Brouwer, while the second pair is suggested by Lukasiewicz.

Also, a widely used conjunctor is $c(b,y) = by$, see [6]. Its adjoint implicator is

$$i(b,x) = \begin{cases} \min\left(1, \frac{x}{b}\right) & b \neq 0, \\ 1 & b = 0. \end{cases}$$

2.1 General definition of fuzzy morphology

There are different ways to define fuzzy morphological operations. An immediate paradigm for defining fuzzy morphological operators is to lift each binary operator to a grey-scale operator by fuzzifying its primitive composing operations. However thus we rarely obtain erosion-dilation adjunctions, which leads to non-idempotent openings and closings. Therefore we use the idea from [3], saying that having an adjoint conjunctor-implicator pair, we can define a fuzzy erosion-dilation adjunction.

So let us consider the universal set E and a class of fuzzy sets $\{A_y, \mid y \in E\}$. Then for any fuzzy subset X of the universal set E, let us define fuzzy dilation and erosion as follows:

$$\delta(X)(x) = \bigvee_{y \in E} c(A_x(y), X(y)), \tag{3}$$

$$\varepsilon(X)(x) = \bigwedge_{y \in E} i(A_y(x), X(y)). \tag{4}$$

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To prove that this pair of operations is an adjunction, let us consider the case $\delta(X) \subseteq Z$ in fuzzy sense, which means that for every $x, y \in E$ $c(A_x(y), X(y)) \leq Z(x)$. Then $X(y) \leq i(A_x(y), Z(x))$ for all $x, y \in E$.

Since $\varepsilon(Z)(y) = \bigwedge_{x\in E} i(A_x(y), Z(x))$, then we have that $X \subseteq \varepsilon(Z)$, which ends the proof.

3. How to define T-invariant and geodesic fuzzy morphological operations?

Let us consider a universal set E. Let also there exists an Abelian group of automorphisms T in $\mathcal{P}(E)$ such that T acts transitevely on the supremum-generating family $l = \{\{e\}|e \in E\}$ as defined previously. In this case, for shortness we shall say that T *acts transitively on* E. Then having an arbitrary fuzzy subset B from E, we can define a family of fuzzy sets in $\{A_y^B \mid y \in E\}$ such as $A_y^B(x) = B(\tau_y^{-1}(x))$. Recall that for any $\tau \in T$ there exists $y \in E$ such that $\tau = \tau_y$, and for any fuzzy subset M we have that $(\tau(M))(x) = M(\tau^{-1}(x))$. Then having in mind Equations 3 and 4 we can define a fuzzy adjunction by the *structuring element* B by:

$$\delta_B(X)(x) = \bigvee_{y\in E} c(A_x^B(y), X(y)), \tag{5}$$

$$\varepsilon_B(X)(x) = \bigwedge_{y\in E} i(A_y^B(x), X(y)). \tag{6}$$

We show that that the upper operations are T-invariant. To prove this statement, following [5], it is sufficient to show that every such erosion commutes with an arbitrary automorphism τ_b for any $b \in E$. Evidently

$$\varepsilon_B(\tau_b(X))(x) = \bigwedge_{y\in E} i(B(\tau_y^{-1}(x)), X(\tau_b^{-1}(y))).$$

suppose that $\tau_b^{-1}(y) = z$, which means that $\tau_y = \tau_z\,\tau_b$. Then

$$\varepsilon_B(\tau_b(X))(x) = \bigwedge_{z\in E} i(B(\tau_z^{-1}(\tau_b^{-1}(x)), X(z)) = \varepsilon_B(X)(\tau_b^{-1}(x)),$$

which simply means that $\varepsilon_B(\tau_b(X)) = \tau_b(\varepsilon_B(X))$, which ends the proof.

Now consider that in E we have a continuous commutative operation $* : E \times E \mapsto E$. Then let us define $\tau_b(x) = b * x$. In the case of Gödel-Brouwer conjunctor-implicator pair the respective dilation has the form

$$(\delta_B(X))(x) \;=\; \bigvee_{y*z=x} \min(X(y), B(z)).$$

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http://urlib.net/dpi.inpe.br/ismm@80/2007/03.19.13.52

Following the *extension principle* (see [6]) for the definition of the operation $X * B$ between the fuzzy sets X and B over the universal set E, it is evident that in this case

$$\delta_B(X) = \delta_X(B) = X * B.$$

In [6] it is proved that

$$[X * B]_\alpha = [X]_\alpha * [B]_\alpha = \{z \in E |\, z = a * b, a \in [X]_\alpha, b \in [B]_\alpha\}.$$

Following [10], let us say that the points $x, y \in E$ are connected in the fuzzy set A if there exists a path Γ from x to y such that

$$\inf_{z \in \Gamma} A(z) \;\geq\; \min(A(x), A(y)).$$

Let now M be a fuzzy subset of the universal set E, which is a numerical metric space. Then if x and y are two points from E which are connected in M, we can define the following *geodesic distance between x and y* [2]:

$$d_M(x, y) \;=\; \frac{\operatorname{len}(x, y)}{C_M(x, y)}, \qquad (7)$$

where $\operatorname{len}(x, y)$ is the length of the shortest continuous path between x and y due to the metric in E, and

$$C_M(x, y) \;=\; \sup_\Gamma \inf\{M(z) |\, z \in \Gamma\}.$$

Here Γ denotes an arbitrary path between x and y in E. Since we are working with almost connected objects, while a point has membership 0 when it is from the background (i.e., it does not belong to any object on the scene), we may suppose that C_M is always positive. The quantity $d_M(x, y)$ satisfies all properties of a metrics except the triangle inequality, so it is not a real distance. However, if M is a crisp set, then it is equal to the classical geodesic distance. Now we can define a fuzzy geodesic ball

$$[B_M(y, r)](x) \;=\; \begin{cases} 1 & \text{if } d_M(x, y) \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

Having in mind the expressions (3)-(4) and (5)-(6) we can define a fuzzy geodesic adjunction $(\mathcal{E}^r_M, \Delta^r_M)$ as:

$$\Delta^r_M(X)(x) \;=\; \bigvee_{y \in E} c[(B_M(x, r))(y), X(y)],$$

$$\mathcal{E}^r_M(X)(x) \;=\; \bigwedge_{y \in E} i[(B_M(y, r))(x), X(y)].$$

Therefore we can define fuzzy geodesic reconstruction and idempotent fuzzy geodesic openings and closings as in the binary case descibed in [13]. An example of the usage of this operation is given in [9].

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http://urlib.net/dpi.inpe.br/ismm@80/2007/03.19.13.52

4. Interval computations and computations with fuzzy numbers

Interval computations are computations using intervals with the aim to guarantee the result in particular in the presence of data uncertainties and rounding errors. Since the α-cuts of the fuzzy numbers are closed intervals, then the interval calculus is essential part of the computations with fuzzy numbers. A fuzzy number is a fuzzy subset of $\mathbb{R}$, i.e. it represents a generalization of a real number r. Any fuzzy number A satisfies the following conditions ([6]):

- $A(x) = 1$ for exactly one x;
- the support of A is bounded;
- the α-cuts of A are closed intervals.

In [8] it is shown that there exists a close relation between interval and morphological operations. Having in mind this relation and our general definition of fuzzy morphological operations, we can express the known arithmetic operations between fuzzy numbers through morphological ones and thus we can define inner operations. As shown in [8], the outer and inner interval operations are related to binary dilations and erosions as follows:

$$A + B = A \oplus B = \delta_A(B) = \delta_B(A),$$
$$A +^- B = A \ominus (-B) \cup B \ominus (-A) = \varepsilon_{-B}(A) \cup \varepsilon_{-A}(B).$$

Now let denote by $F(\mathbb{R})$ the set of fuzzy numbers. Then we can define following operations on them using the *extension principle* [6]:

$$(A + B)(x) = \bigvee_{z+y=x} \min(A(y), B(z));$$

$$(A \times B)(x) = \bigvee_{z.y=x} \min(A(y), B(z));$$

$$(A - B)(x) = \bigvee_{y-z=x} \min(A(y), B(z)) = (A + (-B))(x);$$

$$\frac{A}{B}(x) = \bigvee_{zx=y} \min(A(y), B(z)) = \left(A \times \frac{1}{B}\right)(x).$$

Note that every real number r could be considered as fuzzy number with membership function, which is zero on the whole real line, except in r where it takes value 1.

The sum, the difference and the product of fuzzy numbers are also fuzzy numbers. The division is always possible, however the result is a fuzzy number only when $0 \notin \text{supp}(B)$. In general, the quotient is a fuzzy quantity

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Rio de Janeiro, Brazil, Oct. 10–13, 2007, MCT/INPE, v. 1, p. 39–48.
http://urlib.net/dpi.inpe.br/ismm@80/2007/03.19.13.52

over the real line which support may not be bounded. Also, if A and B are fuzzy numbers then $[A+B]_\alpha = [A]_\alpha + [B]_\alpha$ and $[A \times B]_\alpha = [A]_\alpha \times [B]_\alpha$. Now consider the group of automorhisms $\tau_b(x)$ in $\mathbb{R}$ and the fuzzy operations on $F(\mathbb{R})$ defined by Gödel-Brouwer conjunctor-implicator pair:

$$(\delta_B(A))(x) = \bigvee_{y * z = x} \min(A(y), B(z)),$$

$$(\varepsilon_B(A))(x) = \inf_{y \in \mathbb{R}} \left(h \left(A(y) - B(\tau_x^{-1}(y)) \right) (1 - A(y)) + A(y) \right),$$

where $h(x) = 1$ when $x \geq 0$ and is zero otherwise.

Now it is clear that if $\tau_b(x) = x + b$ and $* = +$ then

$$(\delta_B(A)) \;=\; A + B.$$

We can also define an *inner addition* operation by

$$A +^- B = \varepsilon_{-B}(A) \cup \varepsilon_{-A}(B).$$

If $\tau_b(x) = xb$ for $b \neq 0$ and $y * z = yz$ then

$$(\delta_B(A)) = A \times B.$$

In this case an inner multiplication exists as well:

$$A \times^- B = \varepsilon_{\frac{1}{B}}(A) \cup \varepsilon_{\frac{1}{A}}(B).$$

Note that in this definition we can work with fuzzy numbers which do not contain 0 in their support. It is not difficult to show directly that $A +^- B \subseteq A + B$ and $A \times^- B \subseteq A \times B$.

5. Fuzzy hit- or- miss transform and intuitionistic fuzzy sets

Remind that an intuitionistic fuzzy subset A from the universal set E is characterised by two functions: the degree of membership $\mu_A(x)$ and the degree of nonmembership $\nu_A(x)$. As described in [4], for every point $x \in E$ we have that $\mu_A(x) + \nu_A(x) \leq 1$. Then one can define intersection ot two intuitionistic sets by taking a t-norm Δ of their membership functions for the resulting membership function, and taking the associated s-norm ∇ of their nonmembership functions for the resulting nonmembership functions. Remind that the associated s-norm is defined by $x \nabla y = 1 - ((1-x)\Delta(1-y))$. For the union of two intuitionistic sets we take s-norm for the membership part and the respective t-norm for the nonmembership part.

It is natural to lift to the fuzzy framework the hit-or-miss morphological operator

$$\tilde{\pi}_{A,B}(X) = \varepsilon_A(X) \Delta\, \varepsilon_B(X^c).$$

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http://urlib.net/dpi.inpe.br/ismm@80/2007/03.19.13.52

A key difference with the binary case is that, since A and B are fuzzy sets, we do not assume that $A \cap B = \emptyset$. Note that here we can use any T-invariant fuzzy operations. Further considerations are done in case of usual translation invariance. The experiments indicate that, in the case of noisy images, it is preferable for the structuring elements to be slightly fuzzy, which means that the values of their membership functions have to be close to one in their support. Note that traditional hit-or-miss operation with crisp templates would only mark the objects in the original word but would not in the noisy realizations. Unlike its classical counterpart, the fuzzy hit-or-miss operation always marked the desired objects. To express clearly in a table how a intuitionistic fuzzy set with a finite domain looks like, let us denote by a/b the membership and nonmembership degree of a given element. This means that if we consider an intuitionistic structuring element, then for any pixel x we use the notation $\mu_A(x)/\nu_A(x)$ to show the respective values in the table. If both values are zero, we simply write 0 at the appropriate place in the table. Note, that the origin is located always at the central element of the table. An example of the usage of such "combine" structuring element for a noisy image is given on Figure 1. The element is described on Table 1. The task is to find a 'c'-shaped pattern with a given size on a grey-scale image. The marked 'c'-shape arround the handle (pointed by an arrow) has been detected with degree of truth 0.54. Similar examples for using such patterns, used to detect given characters in a text, can be found in [7].

Table 1. The hit-or-mis structuring element for finding a shape like the letter 'c'.

0	0	0	0.6/0.2	0.6/0.2	0.6/0.2	0	0	0
0	0	0	0.6/0	1/0	0.6/0	0	0	0
0	0	0	0.6/0	0.6/0	0.6/0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0.6/0.2	0.6/0	0.6/0	0	0	0/0.8	0/0.8	0/0.8	0/0.8
0.6/0.2	1/0	0.6/0	0	0/1	0/1	0/1	0/1	0/1
0.6/0.2	0.6/0	0.6/0	0	0	0/0.8	0/0.8	0/0.8	0/0.8
0	0	0	0	0	0	1/0	0	0
0	0	0	0.8/0	0.8/0	0.8/0	0	0	0
0	0	0	0.8/0	1/0	0.8/0	0	0	0
0	0	0	0.6/0.2	0.6/0.2	0.6/0.2	0	0	0

Interesting applications of intuitionistic models in image processing are given in [14]. Further we are going to experiment the fuzzy hit-or-miss transform by intuitionistic elements for finding skeleta by thinning and pseudoconvex hulls by thickenning and to make experiments with intuitionistic elements based on other fuzzy adjoint operations.

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http://urlib.net/dpi.inpe.br/ismm@80/2007/03.19.13.52

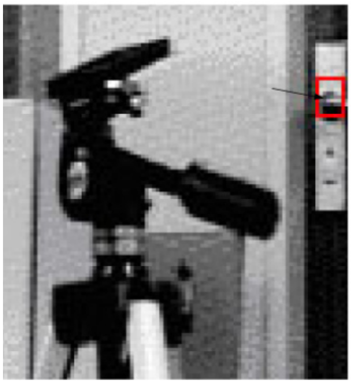

Figure 1. Finding a 'c'-shaped pattern.

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Proceedings of the 8th International Symposium on Mathematical Morphology,
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